On the dynamics of unsteady gravity waves of finite amplitude

Part 1. The elementary interactions

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This paper is concerned with the non-linear interactions between pairs of intersecting gravity wave trains of arbitrary wavelength and direction on the surface of water whose depth is large compared with any of the wavelengths involved. An equation is set up to describe the time history of the Fourier components of the surface displacement in which are retained terms whose magnitude is of order (slope)^2 relative to the linear (first-order) terms. The second-order terms give rise to Fourier components with wave-numbers and frequencies formed by the sums and differences of those of the primary components, and the amplitudes of these secondary components is always bounded in time and small in magnitude. The phase velocity of the secondary components is always different from the phase velocity of a free infinitesimal wave of the same wave-number. However, the third-order terms can give rise to tertiary components whose phase velocity is equal to the phase velocity of a free infinitesimal wave of the same wave-number, and when this condition is satisfied the amplitude of the tertiary component grows linearly with time in a resonant manner, and there is a continuing flux of potential energy from one wave-number to another. The time scale of the growth of the tertiary component is of order of the (-2)-power of the geometric mean of the primary wave slopes times the period of the tertiary wave. The Stokes permanent wave appears as a special case, in which the tertiary wave-number is the same as that of the primary, but its phase is advanced by $\frac{1}{2}\pi$. The energy transfer to the tertiary component in this case is usually interpreted as an increase in the phase velocity of the wave.

The dynamical interactions in water of finite depth are considered briefly, and it is shown that the amplitude of the secondary components becomes large (though bounded in time) as the water depth becomes smaller than the wavelength of the longest primary wave.

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1. Introduction

In this paper, a start is made towards developing a theory to describe the dynamics of the non-linear interactions of a random field of gravity waves of finite amplitude, such as is generated by wind blowing over the sea. In the initial stages of the development of an ocean wave system under the influence of wind, it is probable that the resonance theory (Phillips 1957, 1958*b*) provides the dominant mechanism whereby energy is transferred from the wind to the waves. As the waves develop, however, some 'sheltering' mechanism such as that described by Miles (1957, 1959) for infinitesimal waves, becomes increasingly effective. Also there is the certainty that sooner or later the wave amplitudes will be such that non-linear interactions among the wave components begin to modify the rate of growth of a particular Fourier component by interchange of energy from one component to another. We have, at present, no indication whether this will occur before, or after, the sheltering effect has become dominant nor of the interplay between these two mechanisms. Ultimately, however, if the wind duration is sufficiently great, it appears that a saturation value of the spectral amplitude may be attained (Phillips 1958*a*) beyond which any further development results in an instability of the sea surface and a loss of energy from the wave system by the formation of 'white-caps' or 'white horses'.

It is clear that in order to evaluate the relative roles of the various mechanisms during the generation, a statistical theory of the non-linear interactions will have to be set up. But a number of difficulties present themselves. The most feasible method of approach seems to be in terms of a series development rather analogous to the classical finite amplitude approximations (see Lamb 1932, for example). If we form, say, the covariance of the surface elevation, it might be anticipated that our dynamical equation will contain terms involving third, fourth, and all higher order covariances of the surface displacement, and in fact the dynamical equation for the covariance of any order involves all higher orders. (This is reminiscent of the theory of turbulence—see, for example, Batchelor (1953)—but there the equation for the covariance of order n involves only the order $n+1$ in addition.) It seems evident that, in order to make progress, some approximation must be made, either in truncation of the sequences or on the probability distribution of the surface displacement, one possibility being the joint normal distribution hypothesis which enables the fourth covariances to be related to the second. But until we know something of the essential properties of the interaction process, it is difficult to make an intelligent approximation which will retain these properties in our statistical theory.

The purpose of the present paper is therefore to examine in some detail the 'elementary' dynamical interactions between pairs and among triads of wave components with different wavelengths and directions in order to discover the conditions under which a significant and continuing energy transfer can take place. In the following pages are developed a set of series expansions leading to a dynamical equation involving $dB(\mathbf{k}, t)$, the Fourier–Stieltjes transform of the surface displacement, which is the most accessible experimental observable. We have chosen to use the Fourier–Stieltjes component $dB(\mathbf{k})$ rather than the Fourier coefficient $B(\mathbf{k})$ (which would be slightly more convenient for these elementary interactions) in order that the basic equations be directly applicable to the case of a random sea. Terms are retained to the third order, and we investigate in turn the secondary wave components arising from the second-order interactions of two primary waves and the tertiary wave components from the third-order interactions of either a primary wave with a secondary wave or of three primary waves. We seek the particular types of interaction that are capable of providing a continuing transfer of energy from one component of the wave field to another, so that the latter grows at the expense of the former. With this information there is the possibility of making a meaningful approximation to the statistical problem, fully cognizant of the aspects of the interactions that we neglect and those that we retain.

2. Specification of the motion

Let us suppose that the surface displacement $\xi(\mathbf{x}, t)$ is a stationary random function or a periodic function of position $\mathbf{x} = (x, y)$, but not necessarily of time, so that we can use the Fourier–Stieltjes representation

$$\xi(\mathbf{x}, t) = \int_{\mathbf{k}} dB(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.1}$$

where the integration is over the entire wave-number plane. If $\xi(\mathbf{x}, t)$ is periodic in space, $dB(\mathbf{k}, t)$ degenerates (see Lighthill 1958) to a grid or row of Dirac delta functions multiplied by the element of area in the $\mathbf{k}$-plane. We assume that the water motion is irrotational so that the velocity potential† $\phi(\mathbf{x}, 0, t)$ at the horizontal plane $z = 0$ of the equilibrium free surface is likewise a stationary random (or periodic) function of $\mathbf{x}$. The Fourier–Stieltjes transform $dA(\mathbf{k}, t)$ is given by

$$\phi(\mathbf{x}, 0, t) = \int_{\mathbf{k}} dA(\mathbf{k}, t)\, e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2.2}$$

Our primary concern is with the case of deep water, so that we seek the solution of

$$\nabla^2\phi(\mathbf{x}, z) = 0, \tag{2.3}$$

with the boundary condition (2.2) at $z = 0$ and

$$\phi \to 0 \quad \text{as} \quad z \to \infty. \tag{2.4}$$

This solution is given by

$$\phi(\mathbf{x}, z, t) = \int dA(\mathbf{k}, t)\, e^{-kz}\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2.5}$$

where $k = |\mathbf{k}|$ and z is taken vertically downwards.

The immediate problem is to develop a dynamical equation describing the time history of $dB(\mathbf{k}, t)$ from (2.5) and the kinematic and dynamic boundary conditions on the free surface $z = -\xi$.

The Fourier–Stieltjes transform $dB(\mathbf{k}, t)$ is intimately related to the distribution of potential energy per unit area among the components of the wave field. The mean potential energy per unit projected surface area is (Lamb 1932, p. 428)

$$\begin{aligned} V &= \tfrac{1}{2}\rho g \overline{\xi^2} \\ &= \tfrac{1}{2}\rho g \int \Psi(\mathbf{k})\, d\mathbf{k}, \end{aligned} \tag{2.6}$$

† Or its analytic continuation at points $\mathbf{x}$ where the free surface is below $z = 0$.

where the potential energy spectrum $\Psi(\mathbf{k})$ is given from (2.1) by

$$\Psi(\mathbf{k}) = \frac{\overline{dB(\mathbf{k})\, dB^*(\mathbf{k})}}{dk_1 dk_2}, \tag{2.7}$$

the star denoting a complex conjugate. Non-linear interactions among the components $dB(\mathbf{k})$ can therefore be interpreted in terms of transfer of potential energy from one wave-number to another. The total energy of the motion is, of course, made up of potential and kinetic energy (in equal parts for infinitesimal waves), and there is continuous interchange between the two, but for finite amplitude waves the definition of a meaningful kinetic energy spectrum presents some difficulties. This matter will be taken up in Part II of this paper, since it will not be involved in our present considerations.

3. The equation of motion

(1) The kinematic boundary condition

At the surface $z = -\xi(\mathbf{x}, t)$,

$$\begin{aligned} \frac{D\xi}{Dt} &= \frac{\partial \xi}{\partial t} + (\nabla\phi)_{-\xi} . \nabla\xi \\ &= -(u_z)_{-\xi} = -(\partial\phi/\partial z)_{-\xi}, \end{aligned} \tag{3.1}$$

where ∇ represents the two-dimensional gradient operator $(\partial/\partial x, \partial/\partial y)$ and u_z the downwards vertical velocity component. Thus

$$\frac{\partial \xi}{\partial t} = -\left(\frac{\partial \phi}{\partial z}\right)_{-\xi} - (\nabla\phi)_{-\xi} . \nabla\xi. \tag{3.2}$$

On replacing ξ and ϕ by their Fourier–Stieltjes representations, we obtain

$$\begin{aligned} \int_{\mathbf{k}_0} dB'(\mathbf{k}_0, t) \exp(i\mathbf{k}_0 . \mathbf{x}) &= \int_{\mathbf{k}_0} dA(\mathbf{k}_0, t) \exp(k_0 \xi) \exp(i\mathbf{k}_0 . \mathbf{x}) \\ &+ \int_{\mathbf{k}_0}\int_{\mathbf{k}_1} \mathbf{k}_0 . \mathbf{k}_1 dA(\mathbf{k}_0, t) \exp(k_0\xi)\, dB(\mathbf{k}_1, t) \exp[i(\mathbf{k}_0 + \mathbf{k}_1) . \mathbf{x}], \end{aligned}$$

where the prime denotes a time differentiation. Expanding the factors $\exp(k_0\xi)$ as power series and using (2.1) again, we obtain

$$\begin{aligned} &\int_{\mathbf{k}_0} dB'(\mathbf{k}_0, t) \exp(i\mathbf{k}_0 . \mathbf{x}) \\ &\quad = \int_{\mathbf{k}_0} k_0 dA(\mathbf{k}_0, t) \exp(i\mathbf{k}_0 . \mathbf{x}) \\ &\qquad + \int_{\mathbf{k}_0} \sum_{n=1}^{\infty} (n!)^{-1} \int_{\mathbf{k}_1} \dots \int_{\mathbf{k}_n} k_0^{n+1} dA(\mathbf{k}_0)\, dB(\mathbf{k}_1) \dots dB(\mathbf{k}_n) \\ &\qquad\qquad \times \exp[i(\mathbf{k}_0 + \mathbf{k}_1 + \dots + \mathbf{k}_n) . \mathbf{x}] \\ &\qquad + \int_{\mathbf{k}_0} \sum_{n=1}^{\infty} [(n-1)!]^{-1} \int_{\mathbf{k}_1} \dots \int_{\mathbf{k}_n} \mathbf{k}_0 . \mathbf{k}_1 k_0^{n-1} dA(\mathbf{k}_0)\, dB(\mathbf{k}_1) \dots dB(\mathbf{k}_n) \\ &\qquad\qquad \times \exp[i(\mathbf{k}_0 + \mathbf{k}_1 + \dots + \mathbf{k}_n) . \mathbf{x}]. \end{aligned} \tag{3.3}$$

In the nth term of each of the summations make the change of variable

$$\left.\begin{aligned} \mathbf{k} &= \mathbf{k}_0+\mathbf{k}_1+\ldots+\mathbf{k}_n, \\ \mathbf{k}_1 &= \mathbf{k}_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathbf{k}_n &= \mathbf{k}_n, \end{aligned}\right\} \tag{3.4}$$

and replace $\mathbf{k}_0$ by $\mathbf{k}$ in the single integrals, and it follows that

$$\begin{aligned} dB'(\mathbf{k},t) = k\,dA(\mathbf{k},t)+\sum_{n=1}^{\infty}\int_{\mathbf{k}_1}\ldots\int_{\mathbf{k}_n}\frac{1}{(n-1)!}\,|\mathbf{k}-\mathbf{k}_1-\ldots-\mathbf{k}_n|^{n-1} \\ \times\left\{\frac{1}{n}\,|\mathbf{k}-\mathbf{k}_1-\ldots-\mathbf{k}_n|^2+\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1-\ldots-\mathbf{k}_n)\right\} \\ \times\,dA(\mathbf{k}-\mathbf{k}_1-\ldots-\mathbf{k}_n)\,dB(\mathbf{k}_1)\ldots dB(\mathbf{k}_n), \end{aligned} \tag{3.5}$$

where we assume that the series converges sufficiently rapidly that under at least some conditions (to be specified more precisely later) the first few terms suffice for a good approximation.

The equation (3.5) is the direct Fourier–Stieltjes representation of the kinematic boundary condition (3.2). It will be convenient to express $dA(\mathbf{k},t)$, the Fourier–Stieltjes transform of $\phi(\mathbf{x},0,t)$ in terms of $dB(\mathbf{k},t)$ and its time derivatives. From the first few terms of (3.5),

$$\begin{aligned} k\,dA(\mathbf{k}) = dB'(\mathbf{k})-\int_{\mathbf{k}_1}\{|\mathbf{k}-\mathbf{k}_1|^2+\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1)\}\,dA(\mathbf{k}-\mathbf{k}_1)\,dB(\mathbf{k}_1) \\ -\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|\,\{\tfrac{1}{2}\,|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|^2+\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\} \\ \times\,dA(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,dB(\mathbf{k}_1)\,dB(\mathbf{k}_2), \end{aligned}$$

and by successive substitutions for $dA(\mathbf{k}-\mathbf{k}_1)$ and $dA(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)$ on the right-hand side, we find that, correct to third-order terms,

$$\begin{aligned} dA(\mathbf{k}) = k^{-1}dB'(\mathbf{k})-\int_{\mathbf{k}_1}D_1(\mathbf{k},\mathbf{k}_1)\,dB'(\mathbf{k}-\mathbf{k}_1)\,dB(\mathbf{k}_1) \\ -k^{-1}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2}D_2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\,dB'(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\,dB(\mathbf{k}_1)\,dB(\mathbf{k}_2), \end{aligned} \tag{3.6}$$

where

$$\left.\begin{aligned} D_1(\mathbf{k},\mathbf{k}_1) &= \frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1)}{\mathbf{k}\,|\mathbf{k}-\mathbf{k}_1|}, \\ D_2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) &= \tfrac{1}{2}\,|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|^2+\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \\ &\quad -\frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1)\,(\mathbf{k}-\mathbf{k}_1).(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{|\mathbf{k}-\mathbf{k}_1|\,|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|}. \end{aligned}\right\} \tag{3.7}$$

(ii) *The dynamical boundary condition*

Provided we restrict our attention to wave-numbers less than about $2\,\mathrm{cm}^{-1}$ for which surface tension effects in an air-water system are unimportant, the pressure at the free surface $z = -\xi(\mathbf{x},t)$ is continuous, so that

$$-\frac{p}{\rho} = \left(\frac{\partial\phi}{\partial t}\right)_{-\xi}+\tfrac{1}{2}(\mathbf{u}^2)_{-\xi}+g\xi, \tag{3.8}$$

where p represents the pressure on the surface resulting from the air motion, ρ the water density and g the gravitational acceleration. The Fourier–Stieltjes transform of $p(\mathbf{x},t)$ can be defined by

$$p(\mathbf{x},t) = \int d\varpi(\mathbf{k},t)\, e^{i\mathbf{k}.\mathbf{x}}, \tag{3.9}$$

while the corresponding transform of $\xi(\mathbf{x},t)$ is $dB(\mathbf{k},t)$. It remains to develop expansions, correct to third order, of the other two terms of (3.8).

Let

$$\begin{aligned}\left(\frac{\partial\phi}{\partial t}\right)_{-\xi} &= \int dC(\mathbf{k},t)\, e^{i\mathbf{k}.\mathbf{x}}, \quad \text{say,} \\ &= \int dA'(\mathbf{k},t)\, e^{k\xi}\, e^{i\mathbf{k}.\mathbf{x}}, \end{aligned}\tag{3.10}$$

from (2.5).† The exponential term in (3.10) is expanded, on the supposition that $k\xi$ is small (in mean square), and after changing the variables of integration as in (3.4), we find that

$$\begin{aligned} dC(\mathbf{k},t) &= dA'(\mathbf{k},t) + \int_{\mathbf{k}_1} |\mathbf{k}-\mathbf{k}_1|\, dA'(\mathbf{k}-\mathbf{k}_1)\, dB(\mathbf{k}_1) \\ &\quad + \frac{1}{2}\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} |\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|^2\, dA'(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, dB(\mathbf{k}_1)\, dB(\mathbf{k}_2), \end{aligned}\tag{3.11}$$

correct to the third order The expression (3.6) can now be used to eliminate $dA(\mathbf{k})$ from the right-hand side, which becomes, to this order,

$$\begin{aligned} dC(\mathbf{k},t) &= k^{-1}dB''(\mathbf{k},t) + \int_{\mathbf{k}_1} \{1 - D_1(\mathbf{k},\mathbf{k}_1)\}\, dB''(\mathbf{k}-\mathbf{k}_1)\, dB(\mathbf{k}_1) \\ &\quad - \int_{\mathbf{k}_1} D_1(\mathbf{k},\mathbf{k}_1)\, dB'(\mathbf{k}-\mathbf{k}_1)\, dB'(\mathbf{k}_1) \\ &\quad - \int_{\mathbf{k}_1}\int_{\mathbf{k}_2} E(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\, dB''(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, dB(\mathbf{k}_1)\, dB(\mathbf{k}_2) \\ &\quad - \int_{\mathbf{k}_1}\int_{\mathbf{k}_2} F(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\, dB'(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\, dB'(\mathbf{k}_1)\, dB(\mathbf{k}_2), \end{aligned}\tag{3.12}$$

where

$$\left.\begin{aligned} E(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) &= k^{-1}D_2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) + |\mathbf{k}-\mathbf{k}_1|\, D_1(\mathbf{k}-\mathbf{k}_1,\mathbf{k}_2) - \tfrac{1}{2}|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|, \\ F(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) &= k^{-1}[D_2(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) + D_2(\mathbf{k},\mathbf{k}_2,\mathbf{k}_1)] + |\mathbf{k}-\mathbf{k}_2|\, D_1(\mathbf{k}-\mathbf{k}_2,\mathbf{k}_1), \end{aligned}\right\}\tag{3.13}$$

and the functions D_1 and D_2 are as given previously in (3.7).

Finally, the horizontal velocity components at the surface are given by

$$(u_x, u_y) = \nabla\phi = i\int \mathbf{k}\, dA(\mathbf{k},t)\, e^{k\xi}\, e^{i\mathbf{k}.\mathbf{x}}, \tag{3.14}$$

where $\nabla = (\partial/\partial x_1, \partial/\partial x_2)$, and the vertical velocity component by

$$u_z = \frac{\partial\phi}{\partial z} = \int k\, dA(\mathbf{k},t)\, e^{k\xi}\, e^{i\mathbf{k}.\mathbf{x}}, \tag{3.15}$$

† Note that $\left(\frac{\partial\phi}{\partial t}\right)_{-\xi} \neq \frac{\partial}{\partial t}(\phi)_{-\xi}$ since $\xi = \xi(\mathbf{x},t)$.

so that

$$\mathbf{u}^2 = \int_{\mathbf{k}_0}\int_{\mathbf{k}_1} [k_0 k_1 - \mathbf{k}_0 . \mathbf{k}_1]\, dA(\mathbf{k}_0)\, dA(\mathbf{k}_1) \exp[(k_0 + k_1)\xi] \exp[i(\mathbf{k}_0 + \mathbf{k}_1).\mathbf{x}]. \tag{3.16}$$

The reduction of this expression to a series of terms involving integrations over $dB(\mathbf{k}, t)$ and its time derivatives proceeds along the same lines as before. The three steps are (*a*) expansion of the factor $\exp[(k_0 + k_1)\xi]$ (here two terms suffice to given an expression correct to third order), (*b*) a change of the integration variables to the set (3.4), and (*c*) elimination of $dA(\mathbf{k}, t)$ from the integrals by using (3.6). The reader who follows the calculation through in detail will notice that he obtains two third-order terms involving integrals over

$$dB'(\mathbf{k} - \mathbf{k}_1)\, dB'(\mathbf{k}_1 - \mathbf{k}_2)\, dB(\mathbf{k}_2)$$

and

$$dB'(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, dB'(\mathbf{k}_1)\, dB(\mathbf{k}_2).$$

The first form can be reduced to the second by an obvious further change of variable. The end result of this calculation is that the Fourier–Stieltjes transform of $(\mathbf{u}^2)_{-\xi}$ is found to be

$$\int_{\mathbf{k}_1} \left\{1 - \frac{\mathbf{k}_1 . (\mathbf{k} - \mathbf{k}_1)}{k_1 |\mathbf{k} - \mathbf{k}_1|}\right\} dB'(\mathbf{k} - \mathbf{k}_1)\, dB'(\mathbf{k}_1)$$
$$- \int_{\mathbf{k}_1}\int_{\mathbf{k}_2} G(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, dB'(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, dB'(\mathbf{k}_1)\, dB(\mathbf{k}_2), \tag{3.17}$$

where

$$\begin{aligned} G(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) = {} & |\mathbf{k} - \mathbf{k}_1| \left\{1 - \frac{\mathbf{k}_1 . (\mathbf{k} - \mathbf{k}_1)}{k_1 |\mathbf{k} - \mathbf{k}_1|}\right\} D_1(\mathbf{k} - \mathbf{k}_1, \mathbf{k}_2) \\ & + |\mathbf{k}_1 + \mathbf{k}_2| \left\{1 - \frac{(\mathbf{k}_1 + \mathbf{k}_2) . (\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)}{|\mathbf{k}_1 + \mathbf{k}_2|\, |\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2|}\right\} D_1(\mathbf{k}_1 + \mathbf{k}_2, \mathbf{k}_2) \\ & - \{k_1 + |\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2|\} \left\{1 - \frac{\mathbf{k}_1 . (\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)}{k_1 |\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2|}\right\}. \end{aligned} \tag{3.18}$$

The Fourier–Stieltjes transform of the dynamical boundary condition (3.8) can now be expressed in terms of $d\varpi(\mathbf{k}, t)$ and of $dB(\mathbf{k}, t)$ and its time derivatives. From (3.12) and (3.17), together with the supplementary conditions (3.13) and (3.18), it is found, after a little algebra that

$$\begin{aligned} -\rho^{-1} d\varpi(\mathbf{k}) = {} & k^{-1} dB''(\mathbf{k}) + g\, dB(\mathbf{k}) \\ & + \int_{\mathbf{k}_1} H_1(\mathbf{k}, \mathbf{k}_1)\, dB''(\mathbf{k} - \mathbf{k}_1)\, dB(\mathbf{k}_1) + \int_{\mathbf{k}_1} H_2(\mathbf{k}, \mathbf{k}_1)\, dB'(\mathbf{k} - \mathbf{k}_1)\, dB'(\mathbf{k}_1) \\ & - \int_{\mathbf{k}_1}\int_{\mathbf{k}_2} H_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, dB''(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, dB(\mathbf{k}_1)\, dB(\mathbf{k}_2) \\ & - \int_{\mathbf{k}_1}\int_{\mathbf{k}_2} H_4(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, dB'(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, dB'(\mathbf{k}_1)\, dB(\mathbf{k}_2), \end{aligned} \tag{3.19}$$

where all the transform functions are understood to be functions of time t also, and the primes represent differentiations with respect to t. The functions $H_1, \ldots, H_4$ are given by

$$\begin{aligned} H_1(\mathbf{k}, \mathbf{k}_1) &= 1 - \frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1)}{k\,|\mathbf{k}-\mathbf{k}_1|} \\ &= 1-\cos\theta \\ &= H_1(-\mathbf{k}, -\mathbf{k}_1); \end{aligned} \tag{3.20}$$

$$\begin{aligned} H_2(\mathbf{k}, \mathbf{k}_1) &= \frac{1}{2}\left\{1 - \frac{\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1)}{k_1\,|\mathbf{k}-\mathbf{k}_1|}\right\} - \frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1)}{k\,|\mathbf{k}-\mathbf{k}_1|} \\ &= \tfrac{1}{2}(1-\cos\phi)-\cos\theta \\ &= H_2(-\mathbf{k}, -\mathbf{k}_1), \end{aligned} \tag{3.21}$$

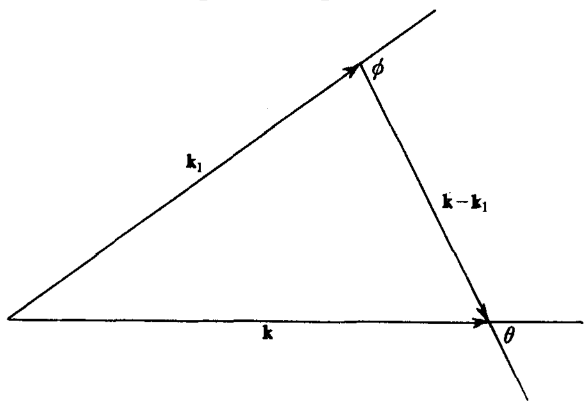

FIGURE 1

the angles θ and ϕ being illustrated in figure 1;

$$\begin{aligned} H_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) &= (2k)^{-1}|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|^2 - \tfrac{1}{2}|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2| + k^{-1}\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \\ &\qquad + |\mathbf{k}-\mathbf{k}_1|\frac{(\mathbf{k}-\mathbf{k}_1).(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{|\mathbf{k}-\mathbf{k}_1|\,|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|}\left\{1-\frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1)}{k\,|\mathbf{k}-\mathbf{k}_1|}\right\}, \\ &= H_3(-\mathbf{k}, -\mathbf{k}_1, -\mathbf{k}_2); \end{aligned} \tag{3.22}$$

$$\begin{aligned} H_4(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) &= \frac{(\mathbf{k}-\mathbf{k}_1).(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{2\,|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|}\left\{1-\frac{\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1)}{k_1\,|\mathbf{k}-\mathbf{k}_1|} - 2\frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1)}{k\,|\mathbf{k}-\mathbf{k}_1|}\right\} \\ &\quad + \frac{(\mathbf{k}_1+\mathbf{k}_2).\mathbf{k}_1}{2k_1}\left\{1-\frac{(\mathbf{k}_1+\mathbf{k}_2).(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{|\mathbf{k}_1+\mathbf{k}_2|\,|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|}\right\} \\ &\quad + \frac{(\mathbf{k}-\mathbf{k}_2).(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|}\left\{1-\frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_2)}{k\,|\mathbf{k}-\mathbf{k}_2|}\right\} + \frac{\mathbf{k}.(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{k} \\ &\quad - \tfrac{1}{2}\{k_1+|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|\}\left\{1-\frac{\mathbf{k}_1.(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)}{k_1\,|\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2|}\right\}, \\ &= H_4(-\mathbf{k}, -\mathbf{k}_1, -\mathbf{k}_2). \end{aligned} \tag{3.23}$$

Equation (3.19) describes the motion of the free surface retaining terms to the third order. The terms involving a single integration describe the second-order interactions and those involving a double integration the third-order interactions. It is clear that these interactions are generally rather complicated algebraically; our aim is to discover the types of interaction that are most significant and which can be described in simple physical terms. In the present paper, we examine the wave dynamics when the air is undisturbed by wind, so that we can take

$$d\varpi(\mathbf{k}, t) = 0$$

in the following sections. We will first examine the properties of the second-order interactions by neglecting the double integral terms in (3.19) and in § 5 we discuss some general properties of the full equation.

4. Secondary wave components

If two primary wave trains of wave-numbers $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ interact, the second-order terms of (3.19) generate components with wave-numbers $\boldsymbol{\kappa}_1 + \boldsymbol{\kappa}_2$ and $\boldsymbol{\kappa}_1 - \boldsymbol{\kappa}_2$. These products of binary interactions between primary components can conveniently be called 'secondary' waves, and similarly the components arising from the interactions among three primary waves or between a primary and a secondary wave can be called 'tertiary' components.

Solutions to (3.19) from given initial conditions will be sought by the usual perturbation technique. The Fourier component $dB(\mathbf{k}, t)$ of the surface displacement is represented as the sum of primary, secondary, and higher components in a manner analogous to the classical Stokes expansion

$$dB(\mathbf{k}, t) = dB_1(\mathbf{k}, t) + dB_2(\mathbf{k}, t) + dB_3(\mathbf{k}, t) + \ldots, \tag{4.1}$$

where $dB_1(\mathbf{k}, t)$ is the solution to the linear equation

$$k^{-1} dB_1''(\mathbf{k}, t) + g\, dB_1(\mathbf{k}, t) = 0, \tag{4.2}$$

and it is supposed that

$$\overline{|\int dB_1(\mathbf{k}, t)|} \gg \overline{|\int dB_2(\mathbf{k}, t)|} \gg \ldots, \tag{4.3}$$

where the overbar indicates the ensemble average. The next approximation, yielding the secondary wave components is made by substituting

$$dB_1(\mathbf{k}, t) + dB_2(\mathbf{k}, t) \quad \text{for} \quad dB(\mathbf{k}, t)$$

into (3.19) and retaining only the lowest (second)-order terms in the non-linear integrals. In virtue of (4.2), we have

$$\begin{aligned} k^{-1} dB_2''(\mathbf{k}) + g\, dB_2(\mathbf{k}) = &-\int_{\mathbf{k}_1} H_1(\mathbf{k}, \mathbf{k}_1)\, dB_1''(\mathbf{k} - \mathbf{k}_1)\, dB_1(\mathbf{k}_1) \\ &-\int_{\mathbf{k}_1} H_2(\mathbf{k}, \mathbf{k}_1)\, dB_1'(\mathbf{k} - \mathbf{k}_1)\, dB_1'(\mathbf{k}_1). \end{aligned} \tag{4.4}$$

The tertiary components are found, likewise, by including the next term in the expansion, and it is found that

$$\begin{aligned} k^{-1} dB_3''(\mathbf{k}) + g\, dB_3(\mathbf{k}) = &-\int_{\mathbf{k}_1} H_1(\mathbf{k}, \mathbf{k}_1)\, [dB_1''(\mathbf{k} - \mathbf{k}_1)\, dB_2(\mathbf{k}_1) + dB_2''(\mathbf{k} - \mathbf{k}_1)\, dB_1(\mathbf{k}_1)] \\ &-\int_{\mathbf{k}_1} H_2(\mathbf{k}, \mathbf{k}_1)\, [dB_1'(\mathbf{k} - \mathbf{k}_1)\, dB_2'(\mathbf{k}_1) + dB_2'(\mathbf{k} - \mathbf{k}_1)\, dB_1'(\mathbf{k}_1)] \\ &+\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} H_3(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, dB_1''(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, dB_1(\mathbf{k}_1)\, dB_1(\mathbf{k}_2) \\ &+\int_{\mathbf{k}_1}\int_{\mathbf{k}_2} H_4(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\, dB_1'(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\, dB_1'(\mathbf{k}_1)\, dB_1(\mathbf{k}_2). \end{aligned} \tag{4.5}$$

Consideration of this last equation is postponed to § 5.

4.1. *Single wave train*

The simplest problem is to calculate the secondary components generated by the interaction of an initial sine wave with itself. Although the result is well known the derivation will be discussed in some detail primarily to illustrate a systematic method of selection of the relevant constituents of the interaction. The analysis of more complicated situations follows exactly the same lines, so that much of the detail can be omitted in later sections.

Suppose that

$$dB_1(\mathbf{k},t) = \tfrac{1}{2}\alpha\{\delta(l-l_0)\exp(-in_0t)+\delta(l+l_0)\exp(in_0t)\}\,\delta(m)\,dl\,dm, \qquad (4.6)$$

where

$$n_0 = (gl_0)^{\frac{1}{2}}, \qquad (4.7)$$

$\delta(l)$ and $\delta(m)$ represent the Dirac delta functions and $\mathbf{k} = (l, m)$. The expression (4.6) satisfies (4.2) and is the Fourier–Stieltjes representation of the primary component

$$\xi_1(x,t) = \alpha\cos(l_0x-n_0t). \qquad (4.8)$$

The secondary components are determined by (4.4). When $dB_1(\mathbf{k},t)$ is of the form (4.6), the integral over the wave-numbers $\mathbf{k}_1$ has contributions only when both $\mathbf{k}_1 = (\pm l_0, 0)$ and $\mathbf{k}-\mathbf{k}_1 = (\pm l_0, 0)$, so that the right-hand side is non-zero only near $\mathbf{k} = (\pm 2l_0, 0)$ and $(0,0)$. The formal process of solution is facilitated by integrating the whole equation over a small range containing in turn each of the singular points of the Dirac delta functions, so that the Fourier–Stieltjes transforms degenerate to the coefficients of a Fourier series. If

$$B_2(\mathbf{K},t) = \int_{K_1-\epsilon}^{K_1+\epsilon}\int_{K_2-\epsilon}^{K_2+\epsilon} dB_2(\mathbf{k},t),$$

where $\mathbf{K} = (K_1, K_2) = (\pm 2l_0, 0)$ or $(0,0)$, equation (4.4) becomes

$$B_2''(\mathbf{K},t)+gKB_2(\mathbf{K},t) = -K\int_{\mathbf{k}}\int_{\mathbf{k}_1}\{H_1(\mathbf{k},\mathbf{k}_1)\,dB_1''(\mathbf{k}-\mathbf{k}_1)\,dB_1(\mathbf{k}_1) + H_2(\mathbf{k},\mathbf{k}_1)\,dB_1'(\mathbf{k}-\mathbf{k}_1)\,dB_1'(\mathbf{k}_1)\}, \qquad (4.9)$$

where the $\mathbf{k}$-integration is between the limits specified above. Substitution of (4.6) into the right-hand side of (4.9) gives rise to eight terms, and it is a simple matter to pick out the ones relevant to each of the three significant values of $\mathbf{K}$.

When $\mathbf{K} = (2l_0, 0)$, we require $\mathbf{k} = (2l_0, 0)$, $\mathbf{k}_1 = (l_0, 0)$, so that from (4.6) and (4.9),

$$\left\{\frac{d^2}{dt^2}+2l_0g\right\}B_2(2l_0,t) = \tfrac{1}{2}\alpha^2n_0^2l_0\exp(-2in_0t)\{H_1(2l_0,l_0)+H_2(2l_0,l_0)\}, \qquad (4.10)$$

where only the x-components of the vector arguments are specified explicitly (the components in the y-direction are all zero). Similarly, when $\mathbf{K} = (-2l_0, 0)$, $\mathbf{k} = (-2l_0, 0)$ and $\mathbf{k}_1 = (-l_0, 0)$ so that

$$\left\{\frac{d^2}{dt^2}+2l_0g\right\}B_2(-2l_0,t) = \tfrac{1}{2}\alpha^2n_0^2l_0\exp(2in_0t)\{H_1(-2l_0,-l_0)+H_2(-2l_0,-l_0)\}. \qquad (4.11)$$

The case $\mathbf{K} = 0$ is trivial, the equation reducing to

$$\frac{d^2}{dt^2} B_2(0,t) = 0.$$

Under the present circumstances, the functions $H_1(\mathbf{k}, \mathbf{k}_1)$ and $H_2(\mathbf{k}, \mathbf{k}_1)$ simplify considerably. Here $\mathbf{k}$ and $\mathbf{k}_1$, are of the forms $(K_0, 0)$ and $(K_1, 0)$ respectively, and from the definitions (3.20) and (3.21),

$$\left.\begin{aligned} H_1(K_0,K_1) = H_1(-K_0,-K_1) &= 0 \quad \text{if} \quad K_0 > K_1, \\ &= 2 \quad \text{if} \quad K_1 > K_0 > 0; \\ H_2(K_0,K_1) = H_2(-K_0,-K_1) &= 0 \quad \text{if} \quad K_0, K_1 \text{ of opposite sign}, \\ &= -1 \quad \text{if} \quad K_0 > K_1 > 0, \\ &= 2 \quad \text{if} \quad K_1 > K_0 > 0. \end{aligned}\right\} \quad (4.12)$$

The equations for the secondary components can thus be combined in the form

$$\left\{\frac{d^2}{dt^2} + 2l_0 g\right\} B_2(\pm 2l_0, t) = -\tfrac{1}{2}\alpha^2 n_0^2 l_0 \exp(\mp 2in_0 t). \quad (4.13)$$

Since the initial wave field was sinusoidal, the initial conditions to be imposed on the component B_2 are that

$$B_2 = B_2' = 0 \quad \text{at} \quad t = 0$$

and the solution is readily found to be

$$\begin{aligned} B_2(\pm 2l_0, t) = \tfrac{1}{4}l_0\alpha^2 \exp(\mp 2in_0 t) + \tfrac{1}{8}(\surd[2]-1)\, l_0\alpha^2 \exp(\pm i\surd[2]\, n_0 t) \\ - \tfrac{1}{8}(\surd[2]+1)\, l_0\alpha^2 \exp(\mp i\surd[2]\, n_0 t), \quad (4.14) \end{aligned}$$

where $n_0 = (gl_0)^{\frac{1}{2}}$. In physical space, the secondary wave components are given by

$$\begin{aligned} \xi_2(x,t) &= B_2(2l_0)\exp(2il_0 x) + B_2(-2l_0)\exp(-2il_0 x) \\ &= \tfrac{1}{2}l_0\alpha^2\cos(2l_0 x - 2n_0 t) + \tfrac{1}{4}(\surd[2]-1)\, l_0\alpha^2 \cos(2l_0 x + \surd[2]\, n_0 t) \\ &\qquad - \tfrac{1}{4}(\surd[2]+1)\, l_0\alpha^2\cos(2l_0 x - \surd[2]\, n_0 t), \quad (4.15) \end{aligned}$$

and the solution to the problem, correct to the second order, is

$$\begin{aligned} \xi(x,t) &= \alpha\{\cos(l_0 x - n_0 t) + \tfrac{1}{2}l_0\alpha\cos 2(l_0 x - n_0 t)\} \\ &\quad + \tfrac{1}{4}l_0\alpha^2\{(\surd[2]-1)\cos(2l_0 x + \surd[2]\, n_0 t) - (\surd[2]+1)\cos(2l_0 x - \surd[2]\, n_0 t)\}. \end{aligned} \quad (4.16)$$

Certain properties of this solution are of interest in relation to the analysis of more complex situations later. The secondary wave components are of two types. The first is a 'bound secondary' component, given by the first term of (4.15), which has half the wavelength and twice the frequency of the primary component, and so travels at the same phase velocity of the primary. It represents a distortion of the initial sine wave; the first two terms of (4.16) specify a Stokes permanent wave to this order. The remaining terms come from the complementary function in the solution to (4.13), and their amplitude depends on the particular initial conditions chosen for $B_2(\pm 2l_0, t)$. They have half the wavelength of the primary wave and $\sqrt{2}$ times the frequency, so that their velocity is that of free infinitesimal waves of the same wave-number. These can be called 'free secondary components', and they are dynamically similar to the primary components but their amplitude is smaller by a factor of order $\alpha l_0 \ll 1$.

A point of particular interest is that the amplitudes of the secondary components are *bounded in time*. From the point of view of potential energy transfer, the second-order interactions in this case merely transfer a pulse of energy to the wave-number $2l_0$, but this does not continue. The Stokes permanent wave and the free secondary waves propagate independently (to this order) and do not interact further.

4.2. *Two intersecting wave trains*

Suppose that at time $t = 0$, the surface displacement and velocity correspond to two intersecting sinusoidal waves with wave-numbers $\mathbf{K}_0 = (l_0, m_0)$ and $\mathbf{K}_1 = (l_1, m_1)$. The primary wave field is thus

$$\xi_1(\mathbf{x}, t) = \alpha \cos(\mathbf{K}_0 . \mathbf{x} - n_0 t) + \beta \cos(\mathbf{K}_1 . \mathbf{x} - n_1 t), \tag{4.17}$$

whose Fourier–Stieltjes representation is given by

$$\begin{aligned} dB_1(\mathbf{k}, t) = {} & \tfrac{1}{2}\alpha\{\delta(\mathbf{k}-\mathbf{K}_0)\exp(-in_0 t) + \delta(\mathbf{k}+\mathbf{K}_0)\exp(in_0 t)\}\,d\mathbf{k} \\ & + \tfrac{1}{2}\beta\{\delta(\mathbf{k}-\mathbf{K}_1)\exp(-in_1 t) + \delta(\mathbf{k}+\mathbf{K}_1)\exp(in_1 t)\}\,d\mathbf{k}, \end{aligned} \tag{4.18}$$

where $\delta(\mathbf{k}-\mathbf{K}_0) = \delta(l-l_0)\,\delta(m-m_0)$ and $d\mathbf{k} = dl\,dm$. Substitution of (4.18) into the linear equation (4.2) determines the frequencies n_0 and n_1 of these primary components:

$$\left.\begin{aligned} n_0^2 &= gK_0 = g\{l_0^2 + m_0^2\}^{\frac{1}{2}}, \\ n_1^2 &= gK_1 = g\{l_1^2 + m_1^2\}^{\frac{1}{2}}. \end{aligned}\right\} \tag{4.19}$$

The substitution of (4.18) into the perturbation integrals of (4.4) for the secondary wave components gives rise to a total of 32 terms, not all of which however are of present interest. Four pairs of terms describe the development of wave-numbers $\pm 2\mathbf{K}_0$ and $\pm 2\mathbf{K}_1$, and the generation of the Stokes waves discussed previously. A total of eight terms give the trivial zero wave-number case. The remaining sixteen gives the contributions at wave-numbers $\pm(\mathbf{K}_0 + \mathbf{K}_1)$ and $\pm(\mathbf{K}_0 - \mathbf{K}_1)$, and these are the ones that we wish to select.

When $\mathbf{K} = (\mathbf{K}_0 - \mathbf{K}_1)$, $\mathbf{k} = (\mathbf{K}_0 - \mathbf{K}_1)$ and $\mathbf{k}_1 = \mathbf{K}_0$ or $-\mathbf{K}_1$. Selecting out the relevant terms and integrating over a small range including the Dirac delta functions to reduce the Fourier–Stieltjes transforms to Fourier coefficients, one finds that

$$\begin{aligned} & \left\{\frac{d^2}{dt^2} + g\,|\mathbf{K}_0 - \mathbf{K}_1|\right\} B_2(\mathbf{K}_0 - \mathbf{K}_1, t) \\ & \quad = \tfrac{1}{4}\,|\mathbf{K}_0 - \mathbf{K}_1|\,\alpha\beta\exp\{-i(n_0 - n_1)t\}\{n_1^2 H_1(\mathbf{K}_0 - \mathbf{K}_1, \mathbf{K}_0) + n_0 n_1 H_2(\mathbf{K}_0 - \mathbf{K}_1, \mathbf{K}_0) \\ & \qquad + n_0^2 H_1(\mathbf{K}_0 - \mathbf{K}_1, -\mathbf{K}_1) + n_0 n_1 H_2(\mathbf{K}_0 - \mathbf{K}_1, -\mathbf{K}_1)\} \qquad (4.20) \\ & \quad = \tfrac{1}{4}\,|\mathbf{K}_0 - \mathbf{K}_1|\,\alpha\beta n_0 n_1 \exp\{-i(n_0 - n_1)t\}\,\Gamma(n_0/n_1, \theta, \phi), \qquad (4.21) \end{aligned}$$

say, where

$$\Gamma\left(\frac{n_0}{n_1}, \theta, \phi\right) = \frac{n_1}{n_0} H_1(\mathbf{K}_0-\mathbf{K}_1, \mathbf{K}_0) + H_2(\mathbf{K}_0-\mathbf{K}_1, \mathbf{K}_0) + \frac{n_0}{n_1} H_1(\mathbf{K}_0-\mathbf{K}_1, -\mathbf{K}_1) + H_2(\mathbf{K}_0-\mathbf{K}_1, -\mathbf{K}_1)$$

$$= \frac{n_1}{n_0} H_1(-\mathbf{K}_0+\mathbf{K}_1, -\mathbf{K}_0) + H_2(-\mathbf{K}_0+\mathbf{K}_1, -\mathbf{K}_0) + \frac{n_0}{n_1} H_1(-\mathbf{K}_0+\mathbf{K}_1, \mathbf{K}_1) + H_2(-\mathbf{K}_0+\mathbf{K}_1, \mathbf{K}_1), \quad (4.22)$$

in virtue of (3.20) and (3.21).

Similarly, when $\mathbf{K} = -(\mathbf{K}_0-\mathbf{K}_1)$, $\mathbf{k} = -(\mathbf{K}_0-\mathbf{K}_1)$ and $\mathbf{k}_1 = -\mathbf{K}_0$ or $\mathbf{K}_1$. Selecting the relevant terms in this case we obtain

$$\left\{\frac{d^2}{dt^2} + g\,|\mathbf{K}_0-\mathbf{K}_1|\right\} B_2\{-(\mathbf{K}_0-\mathbf{K}_1), t\} = \tfrac{1}{4}\,|\mathbf{K}_0-\mathbf{K}_1|\,\alpha\beta n_0 n_1 \Gamma\left(\frac{n_0}{n_1}, \theta, \phi\right) \exp\{i(n_0-n_1)\,t\}, \quad (4.23)$$

the function Γ occurring again by (4.22).

For the other secondary components, when $\mathbf{K} = \pm(\mathbf{K}_0+\mathbf{K}_1)$, $\mathbf{k} = \pm(\mathbf{K}_0+\mathbf{K}_1)$ and $\mathbf{k}_1 = \pm\mathbf{K}_0$ or $\pm\mathbf{K}_1$, the selection procedure gives

$$\left\{\frac{d^2}{dt^2} + g\,|\mathbf{K}_0+\mathbf{K}_1|\right\} B_2\{\pm(\mathbf{K}_0+\mathbf{K}_1), t\} = \tfrac{1}{4}\,|\mathbf{K}_0+\mathbf{K}_1|\,\alpha\beta n_0 n_1 \Delta\left(\frac{n_0}{n_1}, \theta, \phi\right) \exp\{\mp i(n_0+n_1)\,t\}, \quad (4.24)$$

where

$$\Delta\left(\frac{n_0}{n_1}, \theta, \phi\right) = \frac{n_1}{n_0} H_1(\mathbf{K}_0+\mathbf{K}_1, \mathbf{K}_0) + H_2(\mathbf{K}_0+\mathbf{K}_1, \mathbf{K}_0) + \frac{n_0}{n_1} H_1(\mathbf{K}_0+\mathbf{K}_1, \mathbf{K}_1) + H_2(\mathbf{K}_0+\mathbf{K}_1, \mathbf{K}_1) \quad (4.25)$$

is symmetrical with respect to changes in sign of both the vector arguments or with respect to interchange of $\mathbf{K}_0$ and $\mathbf{K}_1$.

The equations (4.21), (4.23) and (4.24) are all of the form

$$B_2'' + \nu^2 B_2 = A\, e^{i\mu t},$$

and the solutions are of the form

$$\left.\begin{aligned} B_2 &= \frac{A}{\nu^2-\mu^2} e^{i\mu t} \quad \text{if} \quad \mu^2 \neq \nu^2, \\ \text{or} \quad B_2 &= -\frac{iA}{2\mu}\, t\, e^{i\mu t} \quad \text{if} \quad \mu^2 = \nu^2. \end{aligned}\right\} \quad (4.26)$$

The former possibility represents the secondary components of bounded amplitude that we have encountered in the previous section. The latter type of solution, however, would represent a secondary wave component whose amplitude increases with time, or, in terms of the potential energy spectrum, it would represent a *continuing* energy transfer from one component of the wave field to another.

The condition that is required for a continuing transfer of potential energy between different wave-numbers is now clear. It is that the perturbation term for a certain wave-number should have a frequency equal to the frequency of *free infinitesimal* surface waves of the same wave-number. In other words, the non-linear perturbation terms represent a moving disturbance that induces a forced motion of the higher order components. If the frequency of the disturbance term of a certain wave-number is equal to the frequency of a free infinitesimal wave of the same wave-number, then a resonance occurs, and the amplitude of the forced, higher order component grows rapidly. This type of resonance is closely analogous to that which occurs when a turbulent wind blows across the water surface (Phillips 1957), and we should perhaps not be surprised to find a similar mechanism here in view of the nature of the governing equations (4.4) and (4.5).

Is there the possibility of such a resonance within this second approximation? Are there real wave-number vectors $\mathbf{K}_0$ and $\mathbf{K}_1$ such that, as required by equations (4.23) and (4.24),

$$g\,|\mathbf{K}_0 \pm \mathbf{K}_1| = (n_0 \pm n_1)^2? \tag{4.27}$$

We have

$$n_0 = (gK_0)^{\frac{1}{2}}, \quad n_1 = (gK_1)^{\frac{1}{2}},$$

and let

$$\mathbf{K}_0 = (l_0, m_0) = (K_0 \sin\gamma_0,\ K_0\cos\gamma_0),$$

$$\mathbf{K}_1 = (l_1, m_1) = (K_1 \sin\gamma_1,\ K_1\cos\gamma_1),$$

so that $\gamma_0 - \gamma_1$ is the angle between the two vectors. Squaring both sides of (4.27), we find that

$$K_0^2 + K_1^2 \pm 2K_0K_1\cos(\gamma_0-\gamma_1) = K_0^2 + K_1^2 + 6K_0K_1 + 4(K_0K_1)^{\frac{1}{2}}(K_0+K_1),$$

or

$$\cos(\gamma_0-\gamma_1) = \pm 3 + 4\left\{\frac{\frac{1}{2}(K_0+K_1)}{(K_0K_1)^{\frac{1}{2}}}\right\}. \tag{4.28}$$

But since $K_0 > 0$, $K_1 > 0$,

$$\tfrac{1}{2}(K_0+K_1) > (K_0K_1)^{\frac{1}{2}},$$

except when $K_0 = K_1$. Clearly, then, there are no real solutions to (4.27) and (4.28) except for the difference wave-number when $K_0 = K_1$, $\gamma_0 - \gamma_1 = 0$ that is, the zero wave-number case of the previous section which is trivial, since the coefficient A in (4.26) is zero. We conclude, then, that such solutions cannot occur in the second-order interactions and that these do not give rise to a continuing energy transfer.

The solutions to (4.21), (4.23) and (4.24), being all of the first type of the set (4.26) can now be written down without difficulty. For the difference wave-numbers $\pm(\mathbf{K}_0 - \mathbf{K}_1)$, the Fourier components of the secondary waves are

$$B_2\{\pm(\mathbf{K}_0-\mathbf{K}_1), t\} = \frac{\alpha\beta n_0 n_1\,|\mathbf{K}_0-\mathbf{K}_1|\,\Gamma(n_0/n_1,\theta,\phi)}{4\{g\,|\mathbf{K}_0-\mathbf{K}_1| - (n_0-n_1)^2\}}\exp\{\mp i(n_0-n_1)t\}, \tag{4.29}$$

and for $\pm(\mathbf{K}_0+\mathbf{K}_1)$,

$$B_2\{\pm(\mathbf{K}_0+\mathbf{K}_1), t\} = \frac{\alpha\beta n_0 n_1\,|\mathbf{K}_0+\mathbf{K}_1|\,\Delta(n_0/n_1,\theta,\phi)}{4\{g\,|\mathbf{K}_0+\mathbf{K}_1| - (n_0+n_1)^2\}}\exp\{\mp i(n_0+n_1)t\}, \tag{4.30}$$

together with, in each case, the *free* secondary components arising from the complementary functions in the equations, whose magnitudes depend on the particular initial conditions imposed and which propagate, to this order, as free infinitesimal waves. The bound second-order perturbation waves are therefore given by

$$\xi_2(\mathbf{x},t) = \tfrac{1}{2}\alpha\beta(K_0K_1)^{\frac{1}{2}}\{C\cos[(\mathbf{K}_0-\mathbf{K}_1).\mathbf{x}-(n_0-n_1)t] + D\cos[(\mathbf{K}_0+\mathbf{K}_1).\mathbf{x}-(n_0+n_1)t]\}, \tag{4.31}$$

where the amplitude functions C and D are given by

$$\left.\begin{aligned} C = C(\mathbf{K}_0,\mathbf{K}_1) &= \Gamma\left(\frac{n_0}{n_1},\theta,\phi\right)\left\{1-\frac{(n_0-n_1)^2}{g\,|\mathbf{K}_0-\mathbf{K}_1|}\right\}^{-1}, \\ D = D(\mathbf{K}_0,\mathbf{K}_1) &= \Delta\left(\frac{n_0}{n_1},\theta,\phi\right)\left\{1-\frac{(n_0+n_1)^2}{g\,|\mathbf{K}_0+\mathbf{K}_1|}\right\}^{-1}. \end{aligned}\right\} \tag{4.32}$$

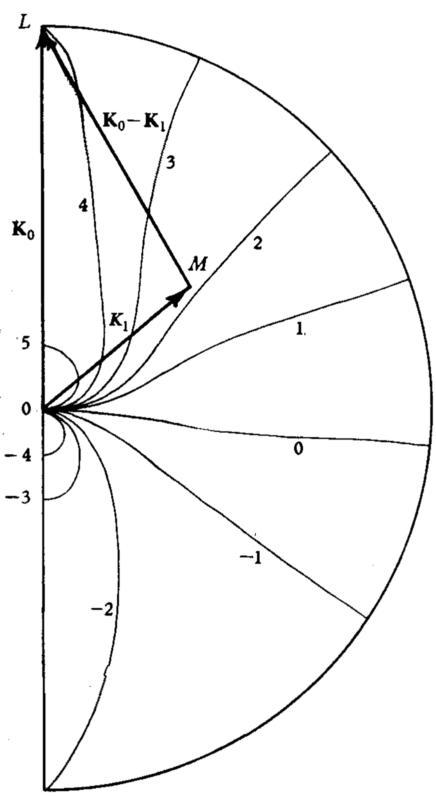

FIGURE 2

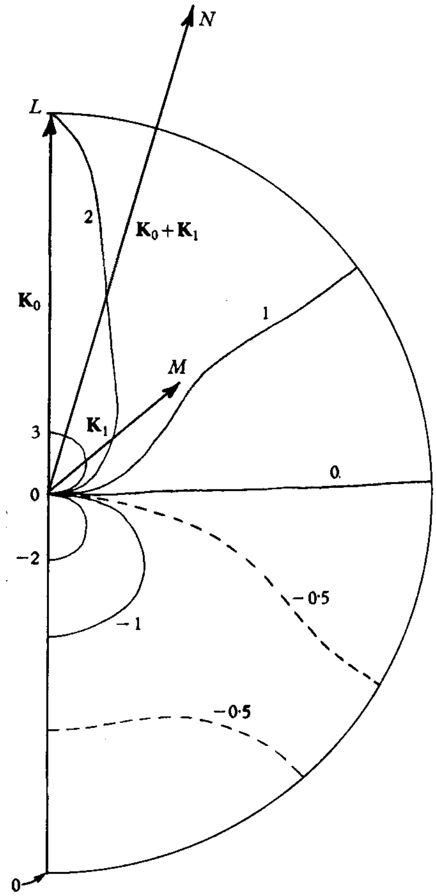

FIGURE 3

FIGURE 2. Contours of the amplitude function C for the difference wave-numbers in second-order interactions. The wave-number $\mathbf{K}_0$, represented by OL, interacts with $\mathbf{K}_1$ (OM) to give the wave-number $\mathbf{K}_0-\mathbf{K}_1$ (ML) whose amplitude is proportional to the value of the function C at the point M. The arrows indicate the directions of propagation of the components.

FIGURE 3. Contours of the amplitude function D for the sum wave-number in second-order interactions. The amplitude of the component with wave-number $\mathbf{K}_0+\mathbf{K}_1$ is given by the value of the function at the point M.

The numerical values of these functions are illustrated in figure 2 and 3. They are symmetrical about the direction of $\mathbf{K}_0$, and have the same values at reciprocal points inside and outside the circle, so that only the region $K_1 \leqslant K_0$ with angles of intersection between 0 and π, need be shown. The important property to observe in this context is that, unless K_1/K_0 is very small (or very large) the amplitude functions are of order unity. The amplitudes of the secondary wave components are thus always small, of order $(\alpha\beta)^{\frac{1}{2}}(\alpha K_0 \beta K_1)^{\frac{1}{2}}$, that is, of the order of the geometric mean of the primary wave amplitudes times the geometric mean of the maximum primary wave slopes.

When the magnitude of one of the primary wave-numbers is much greater than that of the other, K_1/K_0 is very small (or very large), and we are concerned with short waves superimposed on a very much longer wave. The properties of such a system can be derived readily from the expressions given above. This situation has been discussed in detail by Longuet-Higgins & Stewart (1960) who show that the influence of the shorter waves upon the longer can be described in terms of a 'radiation' stress. A number of other special cases are of some interest. When $\mathbf{K}_0 = \mathbf{K}_1$, we have the Stokes permanent wave discussed previously. When $\mathbf{K}_0 = -\mathbf{K}_1$, the second-order approximation to standing waves is obtained. Finally, figures 2 and 3 show that for certain combinations of $\mathbf{K}_0$ and $\mathbf{K}_1$, either C or D is zero, so that the amplitude of the secondary components of either the sum or difference wave-number vanishes.

5. Tertiary wave components

It appears that our search for a mechanism of continuing energy transfer will have to be carried to the next approximation. The algebraic complications become much greater, and careful use of our selection procedure is essential if we wish to avoid being buried under an avalanche of terms. Let us consider first the distortion of an initially sinusoidal wave, to this order of approximation.

5.1. *Single sinusoidal wave train*

We can ignore here any free secondary components, since they propagate as free infinitesimal waves, and their interactions with the primary wave system is of the same nature as discussed in the previous section, except that the products are smaller by an order of magnitude (i.e. by a factor αK_0). The dynamical processes that are new to this situation are the interactions between the primary wave and the bound secondary component, given by the single integral terms on the right-hand side of (4.5), and the triple interactions of the primary wave with itself, given by the double integral terms of that equation. The primary and secondary wave components are given by (4.6) and (4.14), namely

$$\left.\begin{aligned} dB_1(\mathbf{k},t) &= \tfrac{1}{2}\alpha\{\delta(l-l_0)\exp(-in_0t)+\delta(l+l_0)\exp(in_0t)\}\,\delta(m)\,dl\,dm, \\ dB_2(\mathbf{k},t) &= \tfrac{1}{4}l_0\alpha^2\{\delta(l-2l_0)\exp(-2in_0t)+\delta(l+2l_0)\exp(2in_0t)\}\,\delta(m)\,dl\,dm. \end{aligned}\right\} \tag{5.1}$$

The substitution of these expressions into the right-hand side of (4.5), gives rise to contributions when $\mathbf{k} = (\pm l_0, 0)$ and $(\pm 3l_0, 0)$. The various combinations that occur are given in table 1 below, in which the signs associated with an entry in each column is taken as the upper or lower alternative throughout. Thus, in the primary-secondary interactions, the primary wave-number l_0 and the secondary $2l_0$ give rise to the tertiaries $3l_0$ or $-l_0$ (the sum or difference) with the associated time factors $\exp(-3in_0t)$ or $\exp(in_0t)$ respectively. Similarly, in the triple primary interactions, the wave-numbers l_0 and $-l_0$ give rise to a tertiary component of wave-number l_0 or $-l_0$ (in combination with a third primary of wave-number l_0 or $-l_0$) with a time factor of $\exp(-in_0t)$ or $\exp(in_0t)$ respectively.

Primary wave-number	Secondary wave-number	Tertiary wave-number	Time factor
$\pm l_0$	$\pm 2l_0$	$\pm 3l_0$	$\exp(\mp 3in_0t)$
$\pm l_0$	$\pm 2l_0$	$\mp l_0$	$\exp(\pm in_0t)$
$\pm l_0$	$\mp 2l_0$	$\mp l_0$	$\exp(\pm in_0t)$
$\pm l_0$	$\mp 2l_0$	$\pm 3l_0$	$\exp(\mp 3in_0t)$

(*a*) Primary-secondary interactions

k_1	k_2	Tertiary wave-number k $(k-k_1-k_2=\pm l_0)$	Time factor
$\pm l_0$	$\pm l_0$	$\pm 3l_0$	$\exp(\mp 3in_0t)$
$\pm l_0$	$\pm l_0$	$\pm l_0$	$\exp(\mp in_0t)$
$\pm l_0$	$\mp l_0$	$\pm l_0$	$\exp(\mp in_0t)$
$\pm l_0$	$\mp l_0$	$\mp l_0$	$\exp(\pm in_0t)$

(*b*) Triple primary interactions

TABLE 1. Wave-numbers involved in third-order interactions of an initially sinusoidal wave train.

As before, it is convenient to integrate equation (4.5) over a small range of $\mathbf{k}$ containing the wave-number of interest, to obtain the Fourier *coefficients* for this wave-number. The selection of all the relevant terms can now be made, and it is found, for example, that when $\mathbf{k} = (3l_0, 0)$, the primary-secondary interaction term gives

$$\tfrac{1}{8}l_0\alpha^3n_0^2\exp(-3in_0t)\{H_1(3l_0,2l_0)+4H_1(3l_0,l_0)+2H_2(3l_0,2l_0)+2H_2(3l_0,l_0)\}, \quad (5.2)$$

where the numerical coefficients of the H functions arise from the time differentiation of the double frequency secondary components and where only the vectorial components in the x-direction are shown in the arguments. The triple primary interaction term yields

$$-\tfrac{1}{8}\alpha^3n_0^2\exp(-3in_0t)\{H_3(3l_0,l_0,l_0)+H_4(3l_0,l_0,l_0)\}. \quad (5.3)$$

The values of H_1 and H_2 are given, for this simple case, by (4.12), and H_3 and H_4 are evaluated from the expressions (3.22) and (3.23).

The calculations, though still rather tedious, are quite straightforward, and lead to the following expressions for the combined perturbation terms:

$$\left.\begin{aligned} -\tfrac{3}{8}\alpha^3n_0^2l_0\exp(\mp 3in_0t) \quad &\text{when} \quad \mathbf{k}=(\pm 3l_0,0),\\ \tfrac{1}{2}\alpha^3n_0^2l_0\exp(\mp in_0t) \quad &\text{when} \quad \mathbf{k}=(\pm l_0,0). \end{aligned}\right\} \quad (5.4)$$

The dynamical equations for the tertiary components are therefore

$$\left.\begin{aligned}\left(\frac{d^2}{dt^2}+3l_0 g\right)B_3(\pm 3l_0,0) &= -\tfrac{9}{8}\alpha^3 n_0^2 l_0^2 \exp(\mp 3in_0 t),\\ \left(\frac{d^2}{dt^2}+l_0 g\right)B_3(\pm l_0,0) &= \tfrac{1}{2}\alpha^3 n_0^2 l_0^2 \exp(\mp in_0 t).\end{aligned}\right\} \tag{5.5}$$

The first of these equations has the bounded amplitude solution of the first type of the set (4.26), namely

$$B_3(\pm 3l_0,0) = \tfrac{3}{16}\alpha^3 l_0^2 \exp(\mp 3in_0 t), \tag{5.6}$$

together with complementary terms representing free tertiary waves. The second equation of the set (5.5), however, is of the second type of (4.26) and its solution represents a sine wave whose amplitude increases linearly with time:

$$B_3(\pm l_0,0) = \pm\tfrac{1}{4}i\alpha^3 l_0^2(n_0 t)\exp(\mp in_0 t). \tag{5.7}$$

The complete bound tertiary wave components are given by

$$\begin{aligned}\xi_3(x,t) &= B_3(l_0,t)\exp(il_0 x)+B_3(-l_0,t)\exp(-il_0 x)+B_3(3l_0,t)\exp(3il_0 x)\\ &\qquad\qquad +B_3(-3l_0,t)\exp(-3il_0 x)\\ &= \tfrac{1}{2}\alpha^3 l_0^2 n_0 t\sin(l_0 x-n_0 t)+\tfrac{3}{8}\alpha^3 l_0^2\cos 3(l_0 x-n_0 t),\end{aligned} \tag{5.8}$$

and the surface displacement, to the third approximation is

$$\begin{aligned}\xi(x,t) &= \alpha[\cos(l_0 x-n_0 t)+\tfrac{1}{2}\alpha^2 l_0^2 n_0 t\sin(l_0 x-n_0 t)]\\ &\qquad +\tfrac{1}{2}\alpha^2 l_0\cos 2(l_0 x-n_0 t)+\tfrac{3}{8}\alpha^3 l_0^2\cos 3(l_0 x-n_0 t).\end{aligned} \tag{5.9}$$

Here we have our first example of a developing tertiary component in which the frequency of the perturbation term at $\mathbf{k} = (\pm l_0, 0)$ is the same as the frequency of a free infinitesimal surface wave of the same wave-number. This example is, however, rather special in that the wave-number at which the tertiary component grows under this resonance type of interaction is *the same as the primary wave-number*. But the phase of the growing tertiary component is in advance of the phase of the primary wave by $\frac{1}{2}\pi$; this transfer of potential energy to the same wave-number at an earlier phase can be (and usually is) interpreted as an increase in the phase speed of the primary component. As it stands, the solution (5.9) is only valid for $n_0 t \ll (\frac{1}{2}\alpha^2 l_0^2)^{-1}$, after which the developing tertiary wave has an amplitude comparable with that of the primary wave, but the increase in phase velocity is readily found. For this time interval,

$$\tfrac{1}{2}\alpha^3 l_0^2 n_0 t \simeq \sin(\tfrac{1}{2}\alpha^2 l_0^2 n_0 t),$$

so that

$$\cos(l_0 x-n_0 t)+\tfrac{1}{2}\alpha^2 l_0^2 n_0 t\sin(l_0 x-n_0 t) \simeq \cos\{l_0 x-(1+\tfrac{1}{2}\alpha^2 l_0^2)n_0 t\}, \tag{5.10}$$

and the phase velocity c is given by

$$c = c_0(1+\tfrac{1}{2}\alpha^2 l_0^2),$$

where $c_0 = n_0/l_0 = (g/l_0)^{\frac{1}{2}}$, in accord with the classical Stokes expression (see Lamb 1932, § 250). Notice that the interaction time, or the time scale of development of the resonant tertiary component is of the order of (-2)-power of the maximum primary wave slope times the wave period.

5.2. Two intersecting wave trains

Suppose that initially we have two intersecting wave trains with wave-numbers $\mathbf{K}_0 = (l_0, m_0)$ and $\mathbf{K}_1 = (l_1, m_1)$. As in (4.18) the Fourier–Stieltjes components of the primary wave field are given by

$$dB_1(\mathbf{k},t) = \tfrac{1}{2}\alpha\{\delta(\mathbf{k}-\mathbf{K}_0)\exp(-in_0t)+\delta(\mathbf{k}+\mathbf{K}_0)\exp(in_0t)\}\,d\mathbf{k} + \tfrac{1}{2}\beta\{\delta(\mathbf{k}-\mathbf{K}_1)\exp(-in_1t)+\delta(\mathbf{k}+\mathbf{K}_1)\exp(in_1t)\}\,d\mathbf{k}, \quad (5.11)$$

Primary wave-number	Secondary wave-number	Tertiary wave-numbers	
$\pm\mathbf{K}_0$	$\pm 2\mathbf{K}_0$	$\pm 3\mathbf{K}_0,\ \pm\mathbf{K}_0$	(Stokes wave)
$\pm\mathbf{K}_0$	$\pm 2\mathbf{K}_1$	$\pm(\mathbf{K}_0\pm 2\mathbf{K}_1)$	—
$\pm\mathbf{K}_0$	$\pm(\mathbf{K}_0-\mathbf{K}_1)$	$\pm(2\mathbf{K}_0-\mathbf{K}_1),\ \pm\mathbf{K}_1$	—
$\pm\mathbf{K}_0$	$\pm(\mathbf{K}_0+\mathbf{K}_1)$	$\pm(2\mathbf{K}_0+\mathbf{K}_1),\ \pm\mathbf{K}_1$	—

(*a*) Primary-secondary interactions

Primary wave-numbers			Tertiary wave-numbers	
$\pm\mathbf{K}_0$	$+\mathbf{K}_0$	$\pm\mathbf{K}_0$	$\pm 3\mathbf{K}_0,\ \pm\mathbf{K}_0$	(Stokes wave)
$\pm\mathbf{K}_0$	$\pm\mathbf{K}_0$	$\pm\mathbf{K}_1$	$\pm(2\mathbf{K}_0\pm\mathbf{K}_1),\ \pm\mathbf{K}_1$	—
$\pm\mathbf{K}_0$	$\pm\mathbf{K}_1$	$\pm\mathbf{K}_1$	$\pm(\mathbf{K}_0\pm 2\mathbf{K}_1),\ \pm\mathbf{K}_0$	—
$\pm\mathbf{K}_1$	$\pm\mathbf{K}_1$	$\pm\mathbf{K}_1$	$\pm 3\mathbf{K}_1,\ \pm\mathbf{K}_1$	(Stokes wave)

(*b*) Triple primary interactions

TABLE 2. Wave-numbers involved in the third-order interactions of two intersecting wave trains. In part (*a*), further possibilities are obtained by interchanging the subscripts.

and the secondary components are, from (4.14), (4.29) and (4.30),

$$\begin{aligned} dB_2(\mathbf{k},t) = {} & \tfrac{1}{4}K_0\alpha^2\{\delta(\mathbf{k}-2\mathbf{K}_0)\exp(-2in_0t)+\delta(\mathbf{k}+2\mathbf{K}_0)\exp(2in_0t)\}\,d\mathbf{k} \\ & + \tfrac{1}{4}K_1\beta^2\{\delta(\mathbf{k}-2\mathbf{K}_1)\exp(-2in_1t)+\delta(\mathbf{k}+2\mathbf{K}_1)\exp(2in_1t)\}\,d\mathbf{k} \\ & + \tfrac{1}{4}\alpha\beta(K_0K_1)^{\frac{1}{2}}\,C\{\delta(\mathbf{k}-\mathbf{K}_0+\mathbf{K}_1)\exp[-i(n_0-n_1)t] \\ & + \delta(\mathbf{k}+\mathbf{K}_0-\mathbf{K}_1)\exp[i(n_0-n_1)t]\}\,d\mathbf{k} \\ & + \tfrac{1}{4}\alpha\beta(K_0K_1)^{\frac{1}{2}}\,D\{\delta(\mathbf{k}-\mathbf{K}_0-\mathbf{K}_1)\exp[-i(n_0+n_1)t] \\ & + \delta(\mathbf{k}+\mathbf{K}_0+\mathbf{K}_1)\exp[i(n_0+n_1)t]\}\,d\mathbf{k}, \end{aligned} \quad (5.12)$$

where the functions C and D are as given in (4.32). These expressions, if substituted into the right-hand side of (4.5), would give a total of 256 terms, all of which, fortunately, are not of present interest. The various types of interactions can be classified according to table 2 above. The exponential time factors are not indicated, but in every case the exponent has the same structure as the wave-number vector except that the signs are changed. For example, associated with the contribution at wave-number $2\mathbf{K}_0-\mathbf{K}_1$ is the factor

$$\exp\{-i(2n_0-n_1)t\}.$$

The components of particular interest are those with wave-numbers $\pm(2\mathbf{K}_0\pm\mathbf{K}_1)$ (or the symmetrical type $\pm(\mathbf{K}_0\pm 2\mathbf{K}_1)$). We will first investigate whether the resonance interaction is possible apart from the special case discussed in the previous section, which requires the existence of real solutions to the equation

$$g\,|2\mathbf{K}_0 \pm \mathbf{K}_1| = (2n_0 \pm n_1)^2, \tag{5.13}$$

where $n_0 = (gK_0)^{\frac{1}{2}}$, $n_1 = (gK_1)^{\frac{1}{2}}$. Making as before, the substitutions

$$\mathbf{K}_0 = (K_0 \sin\gamma_0, K_0 \cos\gamma_0),$$
$$\mathbf{K}_1 = (K_1 \sin\gamma_1, K_1 \cos\gamma_1),$$

it is found that (5.13) leads to

$$\cos(\gamma_0 - \gamma_1) = \frac{8[\frac{1}{2}(4K_0+K_1)]}{(4K_0K_1)^{\frac{1}{2}}} \pm \left\{\frac{3K_0}{K_1} + 6\right\}. \tag{5.14}$$

Since, for $K_0 > 0$, $K_1 > 0$,

$$\tfrac{1}{2}(4K_0 + K_1) > (4K_0K_1)^{\frac{1}{2}},$$

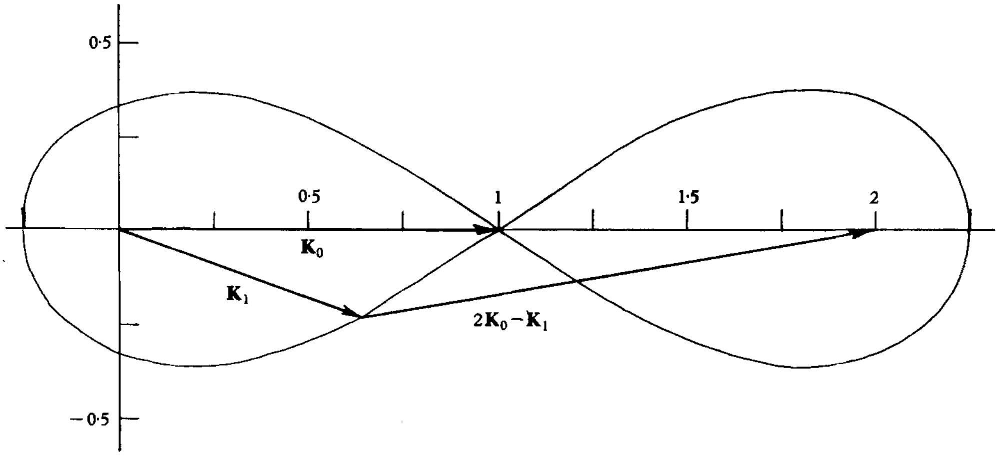

FIGURE 4. The resonance loop for third-order binary interactions. The wave-number $\mathbf{K}_1$ interacts with the bound secondary component associated with $\mathbf{K}_0$ to produce a developing component of wave-number $2\mathbf{K}_0 - \mathbf{K}_1$. The arrows represent the directions of propagation.

it appears that resonant interactions *are* possible, but only for certain pairs of the *difference* wave-numbers $\pm(2\mathbf{K}_0 - \mathbf{K}_1)$ (or for $\pm(\mathbf{K}_0 - 2\mathbf{K}_1)$) for which the negative sign in (5.14) is relevant. Figure 4 represents the function

$$\cos(\gamma_0 - \gamma_1) = \frac{2(\eta+4)}{\sqrt{\eta}} - \frac{3}{\eta} - 6, \tag{5.15}$$

where $\eta = K_1/K_0$, and illustrates the primary wave-numbers $\mathbf{K}_1$ which can produce a resonant interaction with the secondary component of given wave-number $2\mathbf{K}_0$, resulting in a continuing energy transfer to the wave-number $\pm(2\mathbf{K}_0 - \mathbf{K}_1)$. It can readily be confirmed that these resonant tertiary wave-numbers do not coincide with any of the secondary wave-numbers generated, for this would require

$$2\mathbf{K}_0 - \mathbf{K}_1 = \pm 2\mathbf{K}_0,\ \pm 2\mathbf{K}_1,\ \pm(\mathbf{K}_0 + \mathbf{K}_1) \quad \text{or} \quad \pm(\mathbf{K}_0 - \mathbf{K}_1),$$

or equivalently $\quad \mathbf{K}_1 = -2\mathbf{K}_0,\ 0,\ \tfrac{1}{2}\mathbf{K}_0,\ \tfrac{2}{3}\mathbf{K}_0,\ \tfrac{3}{2}\mathbf{K}_0,\ 4\mathbf{K}_0 \quad \text{or} \quad \infty,$

none of which lie on the resonance loop of figure 4. The physical reasons for this type of energy transfer have been described in §4.2. Although it was not possible within the second approximation, it appears that, within the third approximation it is.

Having established the existence of the resonant type of continuing potential energy transfer, it only remains for us to estimate the order of magnitude of the coefficient A in the second equation of the set (4.26) to provide an estimate of the characteristic interaction time, or the development time of the growing tertiary component. Selecting out the relevant contributions as before, it is found that the Fourier coefficient at the wave-number $2\mathbf{K}_0-\mathbf{K}_1$ is given by the solution of

$$\left\{\frac{d^2}{dt^2}+|2\mathbf{K}_0-\mathbf{K}_1|\,g\right\}B_2(2\mathbf{K}_0-\mathbf{K}_1,t) = \tfrac{1}{8}\,|2\mathbf{K}_0-\mathbf{K}_1|\,(K_0K_1)^{\frac{1}{2}}\,n_0n_1\alpha^2\beta f(\eta,\gamma)\exp\{-i(2n_0-n_1)t\}, \qquad (5.16)$$

where $\eta = K_1/K_0$ and γ is the angle between the two interacting wave-numbers. The function $f(\eta,\gamma)$ is given by

$$\begin{aligned} f(\eta,\gamma) = (K_0/K_1)^{\frac{1}{2}}\{&C(\mathbf{K}_0,\mathbf{K}_1)\,[H_1(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0-\mathbf{K}_1)+H_2(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0-\mathbf{K}_1)\\ &+H_1(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0)+H_2(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0)]+4(K_0/K_1)^{\frac{1}{2}}\,H_1(2\mathbf{K}_0-\mathbf{K}_1,-\mathbf{K}_1)\\ &-(K_0K_1)^{-\frac{1}{2}}\,[H_3(2\mathbf{K}_0-\mathbf{K}_1,-\mathbf{K}_1,\mathbf{K}_0)+H_3(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0,-\mathbf{K}_1)\\ &+H_4(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0,-\mathbf{K}_1)]\}\\ &+2(K_0/K_1)^{\frac{1}{2}}\,[H_2(2\mathbf{K}_0-\mathbf{K}_1,2\mathbf{K}_0)+H_2(2\mathbf{K}_0-\mathbf{K}_1,-\mathbf{K}_1)]\\ &-C(\mathbf{K}_0,\mathbf{K}_1)\,[2H_1(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0)+H_2(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0-\mathbf{K}_1)\\ &+H_2(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0)]\\ &-(K_0K_1)^{-\frac{1}{2}}\,[H_4(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0,\mathbf{K}_0)+H_4(2\mathbf{K}_0-\mathbf{K}_1,-\mathbf{K}_1,\mathbf{K}_0)]\\ &+(K_1/K_0)^{\frac{1}{2}}\,\{(K_0/K_1)^{\frac{1}{2}}\,H_1(2\mathbf{K}_0-\mathbf{K}_1,2\mathbf{K}_0)+C(\mathbf{K}_0,\mathbf{K}_1)\,H_1(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0)\\ &-(K_0K_1)^{-\frac{1}{2}}\,H_3(2\mathbf{K}_0-\mathbf{K}_1,\mathbf{K}_0,\mathbf{K}_0)\}, \end{aligned} \qquad (5.17)$$

where the function C is as given in (4.32), and the H functions are specified by (3.20) to (3.23). The equation for the wave-number $-(2\mathbf{K}_0-\mathbf{K}_1)$ is similar (since all the terms in $f(\eta,\gamma)$ are invariant with respect to changes in the signs of all wave-numbers) except that the exponential factor is $\exp\{i(2n_0-n_1)t\}$. When

$$g\,|2\mathbf{K}_0-\mathbf{K}_1| = (2n_0-n_1)^2, \qquad (5.18)$$

the solution to equation (5.16) is

$$B_3\{\pm(2\mathbf{K}_0-\mathbf{K}_1),t\} = \mp\tfrac{1}{16}i\alpha(K_0\alpha)\,(K_1\beta)\,(2n_0-n_1)\,t\,.\,f(\eta,\gamma)\exp\{\mp i(2n_0-n_1)t\}, \qquad (5.19)$$

so that the growing tertiary component is

$$\xi_3(\mathbf{x},t) = \tfrac{1}{8}\alpha f(\eta,\gamma)\,(K_0\alpha)\,(K_1\beta)\,(2n_0-n_1)\,t\,.\sin\{(2\mathbf{K}_0-\mathbf{K}_1)\,.\,\mathbf{x}-(2n_0-n_1)\,t\}. \qquad (5.20)$$

The numerical value of the function $f(\eta,\gamma)$ for wave-numbers such that (5.18) is satisfied could be computed from (5.17), but this would be an extremely tedious operation. It is sufficient for our purpose to know its order of magnitude only. We do already know (either by comparison with the first term on the right-hand side of (5.8), or by direct calculation from (5.17)) that when $\mathbf{K}_0 = \mathbf{K}_1$, $f(\eta,\gamma) = 2$.

Since, on inspection of the forms of the functions $H_1, \ldots, H_4$, it is evident that $f(\eta, \gamma)$ has no singularities, it is safe to assume that this function is of order unity for all wave-number pairs given by the crossed loop of figure 4.

We can therefore conclude that the characteristic development time of the resonant tertiary interactions, or the time for the amplitude to become comparable with that of the primary, is of order

$$T = (K_0\alpha)^{-1}(K_1\beta)^{-1}(2n_0-n_1)^{-1}, \tag{5.21}$$

that is, of order of the wave period of the tertiary component divided by the product of the maximum slopes of the primary waves. As the tertiary component develops, the primary components of course begin to subside as the energy is transferred to the tertiary components. This decay is described by the presence of tertiary terms of wave-number $\pm\mathbf{K}_0$ and $\pm\mathbf{K}_1$ in the last two rows of table 2*a*, and the middle two rows of table 2*b*, and can be calculated in a manner analogous to that above.

In the light of this analysis, the exceptional nature of the Stokes permanent wave is perhaps more evident. Figure 4 shows that this is the *only* situation in which the resonant tertiary wave-number is equal to that of the primary wave, and so provides the only case in which the tertiary interactions can be interpreted as a change in phase velocity of the primary wave. In all other binary interactions, the wave-number of the resonant tertiary component is different from that of either primary, so that a *new* Fourier component of the wave field develops in time.

Finally, it might be remarked that the triple interactions among three different primary waves can also give rise to a developing tertiary component with a new wave-number. The situation discussed above is clearly a special case of this, when two of the primary wave-numbers coalesce. The analysis of the more general case is, however, complicated and little of conceptual value is to be gained from it.

6. Secondary interactions in water at finite depth

We have found that, in water of infinite depth, if the wave-number and frequency of the interaction term correspond to the wave-number and frequency of a *free* infinitesimal wave, then we have a resonant situation in which the amplitude of the interaction wave increases linearly with time. The interaction wave-number is the vector sum or difference of the primary wave-numbers, and the frequency is the sum or difference of the primary frequencies. The simplicity of this criterion and of its physical interpretation makes it possible to anticipate the qualitative nature of the interactions when the water depth is not necessarily large compared with any of the wavelengths involved.

The detailed analysis of this case is very complicated, but the form of the basic equations is not changed. Two primary wave-components of wave-number $\mathbf{K}_0$ and $\mathbf{K}_1$ give rise to an interaction term with wave-numbers $\pm(\mathbf{K}_0+\mathbf{K}_1)$, $\pm(\mathbf{K}_0-\mathbf{K}_1)$ and frequencies n_0+n_1 and n_0-n_1, where

$$\left.\begin{aligned} n_0 &= \{gK_0 \tanh K_0 d\}^{\frac{1}{2}}, \\ n_1 &= \{gK_1 \tanh K_1 d\}^{\frac{1}{2}}, \end{aligned}\right\} \tag{6.1}$$

where d is the mean water depth. If

$$(n_0 \pm n_1)^2 = g\,|\mathbf{K}_0 \pm \mathbf{K}_1|\tanh\,[|\mathbf{K}_0 \pm \mathbf{K}_1|\,d], \tag{6.2}$$

then resonant interactions can occur for the secondary components. The question of the existence or otherwise of real solutions to (6.2) can be most easily established by the following geometrical argument.

In figure 5, the curved line represents the function

$$\left(\frac{d}{g}\right)^{\frac{1}{2}} n = \{Kd \tanh Kd\}^{\frac{1}{2}},$$

specifying the frequency of free infinitesimal surface waves in water of mean depth d as a function of wave-number K. The slope of this curve decreases monotonically with increasing K, except when $Kd \ll 1$, when it is constant. Suppose we

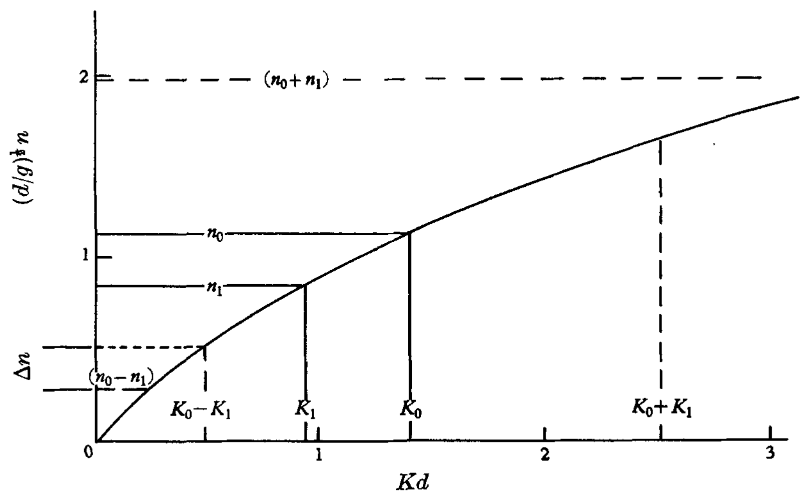

FIGURE 5. Second-order wave-numbers and frequencies in water of finite depth.

have two primary components of wave-numbers $\mathbf{K}_0$ and $\mathbf{K}_1$ and frequencies n_0 and n_1. The interaction frequencies are therefore $n_0 + n_1$ and $n_0 - n_1$ and the magnitudes of the interaction wave-numbers lie within the limits $|K_0 + K_1|$ and $|K_0 - K_1|$, whatever the angle of intersection of the primary waves. If the frequencies $n_0 + n_1$ or $n_0 - n_1$ correspond to wave-numbers lying within that range, then a resonance is possible. It is clear from figure 5, however, that in general this is not so, since the curve is concave downwards and the difference frequency corresponds to a wave-number less than $|K_0 - K_1|$ and the sum frequency to a wave-number greater than $|K_0 + K_1|$. A trivial exception is given when $\mathbf{K}_0 = \mathbf{K}_1$; the difference terms give the zero frequency, zero wave-number case discussed above.

More interesting is the case when for all wave-numbers $\mathbf{K}$ concerned, $Kd \ll 1$. In this shallow water case,† the phase velocity of an infinitesimal wave is in-

† Caution must be exercised lest the small amplitude approximation be pressed too far in shallow water. Stokes (1847) showed that this approximation is valid only when $aK \ll (Kd)^3$, so that when Kd is small, aK is required to be very small indeed.

dependent of the wave-number so that the dispersive nature of the waves is lost and the frequency curve reduces to a straight line (figure 6). It is evident that the sum and difference frequencies correspond to sum and difference wave-numbers at the ends of the permissible range, which are attained when $\mathbf{K}_0$ and $\mathbf{K}_1$ are parallel. Thus, in very shallow water, any pair of parallel wave-numbers will produce resonant interactions in the sense described above. This may also be deduced directly from (6.2), since

$$\tanh[|\mathbf{K}_0 \pm \mathbf{K}_1| d] \simeq |\mathbf{K}_0 + \mathbf{K}_1| d$$

under these circumstances.

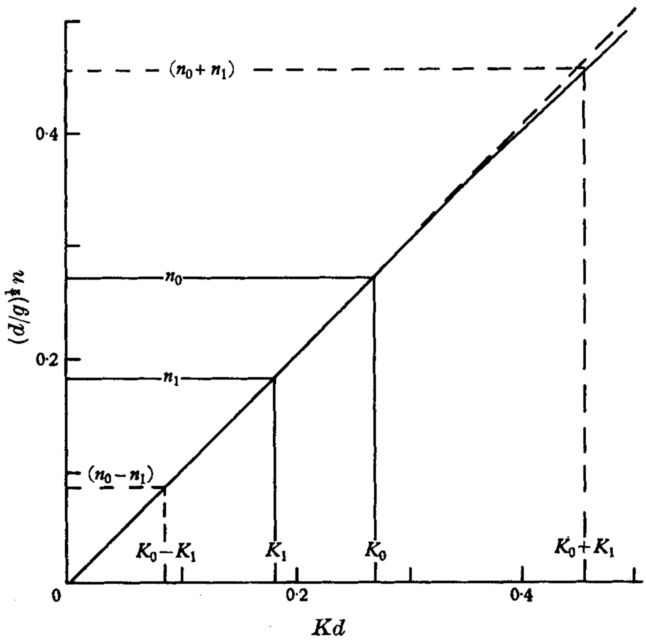

FIGURE 6. Second-order wave-numbers and frequencies in very shallow water.

It may also be interesting to notice that if the difference frequency (say) is only slightly smaller than the frequency of a free surface wave of the difference wave-number, then there is no resonance but, for the first equation of the set (4.26), the amplitude of the bound secondary component is inversely proportional to this frequency difference (shown as Δn in figure 5). If $|\mathbf{K}_0 - \mathbf{K}_1| d$ is of order unity or a little less, Δn may be small, so that the amplitude of the secondary waves, though bounded in time, may be large and the second-order interactions may assume considerable dynamical significance. The tendency for waves to break over shoals, indicating appreciable non-linear interaction, is consistent with this general conclusion.

7. Conclusions

The principal result of this analysis is the demonstration that, although the tertiary interactions among wave components are given by a perturbation term that is algebraically smaller than that representing the secondary interactions, their cumulative dynamical effect is much more profound because of the existence of resonant wave-numbers whose amplitude grows with time. This implies that, in the development of a dynamical theory to describe a finite amplitude random gravity wave field, the tertiary interactions are essential, and any theory in which they are neglected will ignore the dominant mechanism of energy transfer among the wave components.

The results, however, do have immediate relevance to the question of the interaction between a swell and a local storm. Suppose that swell with wave-number $\mathbf{K}_0$ (figure 4) is generated by a distant disturbance on the ocean and passes through a storm area. If the wave-numbers $\mathbf{K}_1$ of the locally generated waves lie along the crossed loop of figure 4, then appreciable components with wave-number $2\mathbf{K}_0 - \mathbf{K}_1$ may be generated by the resonant interaction, and would appear as a 'ghost' to an observation station. The wave-number and direction of this ghost component may be very different from the dominant contributions to the storm or the swell, and should be readily distinguishable from components of the swell scattered by the turbulence in the water near the storm centre (Phillips 1959).

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